An introduction to planar complexes

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1. An introduction to planar complexes

This article introduces planar complexes, which are remotely inspired by simplicial complexes, but used (primarily) to incrementally construct planar embeddings of planar graphs.

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Definition 1.1: The **tuple space** of a set X is named $\mathrm{Tup}(X)$ and defined by:

$$\mathrm{Tup}(X) := \{(t_i)_{i=1,\dots,n} : n \in \mathbb{N}_0, t_i \in X\ \forall i = 1, \dots, n\}$$

■

Definition 1.2: The length of a tuple $t = (t_i)_{i=1,\dots,n}$ is named $|t| := n$. ■

Definition 1.3: Given a tuple $t = (t_i)_{i=1,\dots,n} = (t_1, \dots, t_n)$ with $n \in \mathbb{N}_0$, we define the **reversal** of t, named $\mathrm{rev}(t)$ and defined by:

$$\mathrm{rev}(t) := (t_n, \dots, t_1) = (t_{n-i+1})_{i=1,\dots,n}$$

■

Definition 1.4: Given a tuple $t = (t_i)_{i=1,\dots,n} = (t_1, \dots, t_n)$ with $n \in \mathbb{N}_0$, we define the **rotation** of t by $j \in \mathbb{Z}_n$ (the whole numbers modulo n), named $\mathrm{rot}_j(t)$ and defined by:

$$\mathrm{rot}_j(t) := \left(t_{i+j}\right)_{i=1,\dots,n} \quad \text{with}\ t_k := t_{k-n}\ \forall k = n+1, \dots, 2n$$

In case of $n = 0$, the rotation maps a singleton set to a singleton set, and is uniquely defined by that. ■

Definition 1.5: The **cyclic tuple space** of a set X is named $\mathrm{CycTup}(X)$ and defined by:

$$\mathrm{CycTup}(X) := \mathrm{Tup}(X)_{/\sim}\ \text{with}\ s \sim t \Leftrightarrow \exists \delta(s,t) \in \{0, \dots, n-1\} : s = \mathrm{rot}_{\delta(s,t)}(t)$$

■

Lemma 1.6: The $\sim$ from the definition of the *cyclic tuple space* is indeed an equivalence relation.

Proof: Reflexivity follows by choosing $\delta(s,s) = 0$, and symmetricity by $\delta_{t,s} = -\delta_{s,t}$. Transitivity follows by $\delta(r,s) + \delta(s,t) = \delta(r,t)$. □

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Definition 1.7: The **boolean stack** $\mathrm{BoolStack}(X)$ of a set X is a non-commutative group, and the embedding of a tuple $t \in \mathrm{Tup}(X)$ into $\mathrm{BoolStack}(X)$ are given by:

$$\mathrm{BoolStack}(X) := \langle X : x^2 \ \forall x \in X \rangle$$
$$\mathrm{BoolStack}(t) := \prod_{i=1}^{|t|} t_i \in \mathrm{BoolStack}(X)$$

■

Corollary 1.8: For every tuple $t \in \mathrm{Tup}(X)$: $\mathrm{BoolStack}(t)\,\mathrm{BoolStack}(\mathrm{rev}(t)) = 1$. ■

Definition 1.9: An **independence system** is a pair $(E, \mathcal{F})$, where E (ground set) is a **finite** set (at least in the cases we consider), $\mathcal{F} \subseteq 2^E$ (independent sets), and such that the following holds:

- $\varnothing \in \mathcal{F}$ and
- $B \subseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$.

Cardinality-maximal independent subsets are called bases. The set of all bases of $(E, \mathcal{F})$ is called a **basis system**. ■

Definition 1.10: A **planar complex** is a tuple (V, E, P, a), such that all of the following holds:

$\varnothing \neq V$, E and P are sets, and their elements are called vertices, elements and paths, respectively. (E, P) is an *independence system.*

$a : V \times V \to \mathrm{Tup}(P)$ is a function with the following anti-symmetry property:

$$\begin{aligned} a(v,u) &= \mathrm{rev}(a(u,v)) \quad && \forall u, v \in V \\ a(u,u) &= () && \forall u \in V \end{aligned}$$

A path $p \in P$ is called an **minpath** iff $|p| = 1$, and a **maxpath** iff

$$p \nsubseteq q \quad \forall q \in P \setminus \{p\}$$

For every *minpath* $p \in P$, the following has to hold:

$$p \in a(u,v) \Rightarrow p = a(u,v)_1 \vee p = a(v,u)_1$$

The **terminated stack** $\mathrm{TermStack}(P)$ is similarly to the *boolean stack* defined as a non-commutative group by:

$$\mathrm{TermStack}(P) := \left\langle P : \begin{matrix} \forall p \in P \text{ maxpath} : p^2, \\ \forall e \in E \\ p \cup \{e\} = q \in P \text{ with } p \neq \varnothing : pqp^{-1} \end{matrix} \right\rangle$$

The embedding of a tuple in a *terminated stack* is defined similarly as for the *boolean stack.*

The **planar arrangement** $\mathrm{planarr}(v)$ around $v \in V$ is a cyclic tuple:

$$\exists \pi_v : (V \setminus \{v\}) \to (V \setminus \{v\}) \text{ single permutation cycle :}$$
$$\mathrm{CycTup}(P) \ni \mathrm{planarr}(v) := \mathrm{concat}_{i=0}^{|V|-2}\, a(v, \pi_i u) \text{ for some arbitrary } u \in V$$
$$\text{for which we require that } \mathrm{TermStack}(\mathrm{planarr}(v)) = 1$$

In case of $V = \{v\}$, we define $\mathrm{planarr}(v) := ()$. ■

TODO: how does one split a *planar complex* at a given vertex and position inside the associated *planar arrangement*?

Bibliography