Diagonal taxicab norm

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2. Introduction

Let $\mathbb{K}$ be an arbitrary field with absolute value function $|\bullet| : \mathbb{K} \to \mathbb{R}_{\geq 0}$.

Definition 2.1: Given a $\mathbb{K}$-vector space X, a **norm** on X is a function $p : X \to \mathbb{R}_{\geq 0}$ with the following properties:

- Subadditivity: $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
- Absolute homogeneity: $p(sx) = |s|\ p(x)$ for all $x \in X, s \in \mathbb{K}$,
- Positive definitivity: For all $x \in X$, if $p(x) = 0$, then $x = 0$.

■

3. The naïve diagonal taxicab norm

Definition 3.1: Let $n \in \mathbb{N}$. We define the **naïve diagonal taxicab norm** as:

$$\begin{aligned} &\|\bullet\|_{\mathrm{NDTC}} : \mathbb{K}^n \to \mathbb{R}_{\geq 0}, \\ &\quad (x_i)_{i=1}^n \mapsto \left(\sum_{i=1}^n |x_i|\right) + \left(\sqrt{n} - n\right) \min_{i=1}^n |x_i| \end{aligned}$$

■

Proof: $\|\bullet\|_{\mathrm{NDTC}}$ is indeed a norm because all of the following holds: Let $x, y \in \mathbb{K}^n$ and $s \in \mathbb{K}$. Then:

In general, the following norm bounds hold (whereby $\|\bullet\|_1 : \mathbb{K}^n \to \mathbb{R}_{\geq 0}$ is the taxicab norm):

$$\frac{\sqrt{n}}{n}\|x\|_1 \quad = \quad \frac{1}{\sqrt{n}}\|x\|_1 \quad \leq \quad \|x\|_{\mathrm{NDTC}} \quad \leq \quad \|x\|_1$$

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Well-definedness:

$$\sum_{i=1}^{n} |x_i| \quad \geq \quad n \min_{i=1}^{n} |x_i| \quad \geq \quad \left(n - \sqrt{n}\right) \min_{i=1}^{n} |x_i|$$

Subadditivity:

$$\begin{aligned}
\|x + y\|_{\text{NDTC}} &= \left(\sum_{i=1}^{n} |x_i + y_i|\right) + \left(\sqrt{n} - n\right) \min_{i=1}^{n} |x_i + y_i| \\
&\leq \left(\sum_{i=1}^{n} (|x_i| + |y_i|)\right) + \left(\sqrt{n} - n\right) \min_{i=1}^{n} (|x_i| + |y_i|) \\
&\leq \left(\sum_{i=1}^{n} (|x_i| + |y_i|)\right) + \left(\sqrt{n} - n\right) \left(\min_{i=1}^{n} |x_i| + \min_{j=1}^{n} |y_j|\right) \\
&= \|x\|_{\text{NDTC}} + \|y\|_{\text{NDTC}}
\end{aligned}$$

Note that the $\sum$ in the definition doesn't need any special considerations, and we want the min to get smaller in the split case because it gets multiplied with $\sqrt{n} - n \leq 0$. The transformation in the second line warrants special attention because there the values to minimize over actually get larger, but they only get larger in a sense bounded by the the smallest entry of $\sum$ in the first term.

Absolute homogeneity:

$$\begin{aligned}
\|sx\|_{\text{NDTC}} &= \left(\sum_{i=1}^{n} |sx_i|\right) + \left(\sqrt{n} - n\right) \min_{i=1}^{n} |sx_i| \\
&= \left(\sum_{i=1}^{n} |s||x_i|\right) + \left(\sqrt{n} - n\right) \min_{i=1}^{n} |s||x_i| \\
&= |s|\|x\|_{\text{NDTC}}
\end{aligned}$$

Positive definitivity:

$$\begin{aligned}
0 \overset{!}{=} \|x\|_{\text{NDTC}} &= \left(\sum_{i=1}^{n} |x_i|\right) + \left(\sqrt{n} - n\right) \min_{i=1}^{n} |x_i| \geq 0 \\
\Rightarrow 0 &= |x_i| \quad \forall i = 1, ..., n \\
\Rightarrow 0 &= x
\end{aligned}$$

□

4. Adaptive diagonal taxicab norm

The norm defined in the previous section has a big problem: It has a too strong dependency on the dimension on $n \in \mathbb{N}$, and for differing dimensions $n, m \in \mathbb{N}$, they don't have much in common. The "scaling" of the norm along the dimension only admits going through the center of a (potentially "higher") cube. It doesn't achieve the goal of allowing arbitrary 45° paths for $n > 2$.

For a vector $(x_1, x_2) \in \mathbb{K}^2$, define

$$
\begin{aligned}
|x| &:= (|x_1|, |x_2|) \\
|x|_{\min} &:= \min\{|x_1|, |x_2|\} \\
|x|_{\max} &:= \max\{|x_1|, |x_2|\}
\end{aligned}
$$

For reference, the **naïve diagonal taxicab norm** of the previous section with $n = 2$ fixed is:

$$
\begin{aligned}
\|\bullet\|_{\mathrm{NDTC}} &: \mathbb{K}^2 \to \mathbb{R}_{\geq 0}, \\
(x_1, x_2) &\mapsto |x_1| + |x_2| + \left(\sqrt{2} - 2\right) \min\{|x_1|, |x_2|\} \\
&= |x|_{\min} + |x|_{\max} + \left(\sqrt{2} - 2\right)|x|_{\min} \\
&= |x|_{\max} + \left(\sqrt{2} - 1\right)|x|_{\min} \\
&= (|x|_{\max} - |x|_{\min}) + \sqrt{2}|x|_{\min}
\end{aligned}
$$

Therefore, we want to also explore a different choice: Also allowing to go through the center of an arbitrary lower-dimensional cube. It should coincide with the norm of the previous section for $n \in \{1, 2\}$.

Definition 4.1: We define the property of a function or sequence $s : X \to \mathbb{R}$ being a **monotonically decreasing and non-negative** by fulfilling all of the following:

- For $i, j \in X$ with $i < j$, we have that $s(i) \geq s(j)$, and
- For $i \in X$, we have that $s(i) \geq 0$.

■

We embed the tuples $(x_i)_{i=1}^{n} \subset \mathbb{C}$ of finite length $n \in \mathbb{N}$ into the space of functions with that property by taking the absolute value of each x_i, and then sorting them by decreasing (absolute) value.

Definition 4.2: On the space of monotonically decreasing, non-negative sequences $s : \mathbb{N} \to \mathbb{R}_{\geq 0}$, such that $\lim_{k \to \infty} s(k) = 0$, we define the following **fundamental diagonal taxicab norm**:

$$
\|s\|_{\mathrm{FDTC}} := \sum_{k=1}^{\infty} \sqrt{k} \cdot (s(k) - s(k+1))
$$

Similarly, we define this inthe same way for tuples $(x_i)_{i=1}^{n}$ of finite length $n \in \mathbb{N}$ by embedding it in the space of these sequences, and extending them into infinite sequences by assigning $|x|_i \leftarrow 0$ for all $i > n$.

For simplicity, we consider only sequences for which this “norm” is finite / converges. ■

Proof: Let $r, s : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be monotonically decreasing, non-negative sequences. Let $|\lambda| \in \mathbb{R}_{\geq 0}$.

Linearity:

$$
\begin{aligned}
\|r + s\|_{\mathrm{FDTC}} &= \sum_{k=1}^{\infty} \sqrt{k} \cdot (r(k) + s(k) - (k+1) - s(k+1)) \\
&= \|r\|_{\mathrm{FDTC}} + \|s\|_{\mathrm{FDTC}} \\
\||\lambda| s\|_{\mathrm{FDTC}} &= \sum_{k=1}^{\infty} \sqrt{k} \cdot (|\lambda| s(k) - |\lambda| s(k+1)) \\
&= |\lambda| \|s\|_{\mathrm{FDTC}}
\end{aligned}
$$

Positive definitivity:

$$\begin{aligned} & \|s\|_{\text{FDTC}} = 0 \\ \Longrightarrow \quad & \forall k = 1, ..., \infty : \quad s(k) = s(k+1) \\ \overset{\text{by induction on } k}{\Longrightarrow} \quad & s \equiv s(1) \\ \overset{\text{by convergence to } 0}{\Longrightarrow} \quad & s \equiv 0 \end{aligned}$$

□

For $n = 2$, we have the following:

Theorem 4.3: Let $x_1, x_2 \in \mathbb{K}$. Then $\|(x_1, x_2)\|_{\text{NDTC}} = \|(x_1, x_2)\|_{\text{FDTC}}$.

Proof: As stated at the beginning of this section, the left-hand side is equal to:

$$(|x|_{\max} - |x|_{\min}) + \sqrt{2}|x|_{\min}$$

and the right-hand side is equal to:

$$\sqrt{1}(|x|_{\max} - |x|_{\min}) + \sqrt{2}(|x|_{\min} - 0) + \sum_{k=3}^{\infty} 0 \cdot \sqrt{k}$$

which are equal. □

There is a relation to the forward difference operator:

Definition 4.4: Let $(X, +)$ be a commutative monoid, and $h \in X$ be a step-size (for $X = \mathbb{N}, \mathbb{N}_0, \mathbb{Z}$, we assume $h = 1$ unless stated otherwise). Let $(Y, +)$ be an abelian additive group with inverse $-$. Then we have the **forward difference operator**

$$\begin{aligned} \Delta_h : (X \to Y) \to (X \to Y) \\ \Delta_h[f] \coloneqq X \ni x \mapsto f(x+h) - f(x) \quad \forall f : X \to Y \end{aligned}$$

■

Remark 4.5: We can thus alternatively express/define this *fundamental diagonal taxicab norm* as follows:

$$\|s\|_{\text{FDTC}} \coloneqq \sum_{k=1}^{\infty} (-\Delta_1[s](k))\sqrt{k}$$

■

In order to continue, we need some monotonicity properties:

Lemma 4.6: The functions

$$c : \mathbb{R}_{\geq 1} \to \mathbb{R}_{>0}, x \mapsto \sqrt{x} - \sqrt{x-1}$$
$$c' : \mathbb{R}_{\geq 1} \to \mathbb{R}_{<0}, x \mapsto \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x-1}}}{2}$$
$$c'' : \mathbb{R}_{\geq 1} \to \mathbb{R}_{>0}, x \mapsto \frac{-x^{-\frac{3}{2}} + (x-1)^{-\frac{3}{2}}}{4}$$

is convex, non-negative and monotonically decreasing.

Proof: Non-negativity is obvious. Convexity and monotonically decreasing follows from the ranges of the first and second derivative. □

Lemma 4.7: Let $X \subseteq \mathbb{R}$. For a monotonically decreasing function $c : X \to \mathbb{R}$ the following holds for all $x, y \in \mathbb{R}, k, l \in X$:

$$x \geq y, k \leq l \Rightarrow c(k)x + c(l)y \geq c(l)x + c(k)y$$

Proof: Let $c : X \to \mathbb{R}$ be monotonically decreasing. Let $k, l \in X, k \leq l$ and $x, y \in \mathbb{R}, x \geq y$.

$$\begin{aligned} & c(k) \geq c(l) && \text{because } c \text{ is monotonically decreasing} \\ \Leftrightarrow\ & c(k)(x-y) \geq c(l)(x-y) && \text{because } x \geq y \\ \Leftrightarrow\ & c(k)x + c(l)y \geq c(l)x + c(k)y \end{aligned}$$

□

Definition 4.8: On the space of bounded sequences $f : \mathbb{N} \to \mathbb{K}$, such that a permutation $\pi : \mathbb{N} \to \mathbb{N}$ exists with

$$\begin{aligned} |f(\pi(k))| &\geq |f(\pi(k+1))| \quad \forall k \in \mathbb{N}, \\ 0 &= \lim_{k \to \infty} |f(\pi(k))| \end{aligned}$$

we define the **diagonal taxicab norm**:

$$\begin{aligned} \|f\|_{\mathrm{DTC}} &:= \sum_{k=1}^{\infty} \sqrt{k} \cdot (|f(\pi(k))| - |f(\pi(k+1))|) \\ &= \sum_{k=1}^{\infty} \left(\sqrt{k} - \sqrt{k-1}\right) |f(\pi(k))| \end{aligned}$$

Similarly, we define this in the same way for tuples $(x_k)_{k=1}^{n}$ of finite length $n \in \mathbb{N}$ by embedding it in the space of these sequences, and extending them into infinite sequences by assigning $x_k \leftarrow 0$ for all $k > n$.

For simplicity, we consider only sequences for which this “norm” is finite / converges. ■

Proof: Let $f : \mathbb{N} \to \mathbb{K}$ be in the given space above with corresponding $\pi_f : \mathbb{N} \to \mathbb{N}$. W.l.o.g. assume that π_f is the identity. The two given equation for $\|f\|_{\mathrm{DTC}}$ are equal via:

$$
\begin{aligned}
\sum_{k=1}^{\infty} \sqrt{k} \cdot (f(k) - f(k+1)) &= \sum_{k=1}^{\infty} \sqrt{k} f(k) - \sum_{k=0}^{\infty} \sqrt{k} f(k+1) \\
&= \sum_{k=1}^{\infty} \sqrt{k} f(k) - \sum_{k=1}^{\infty} \sqrt{k-1} f(k) = \sum_{k=1}^{\infty} \left(\sqrt{k} - \sqrt{k-1}\right) f(k)
\end{aligned}
$$

□

Theorem 4.9: The *diagonal taxicab norm* $\|\bullet\|_{\mathrm{DTC}}$ is indeed a norm.

Proof: Let $f, g : \mathbb{N} \to \mathbb{K}$ be in the given space above with corresponding $\pi_f, \pi_g : \mathbb{N} \to \mathbb{N}$. W.l.o.g. assume that π_f, π_g are identity functions. Let $\pi_{f+g} : \mathbb{N} \to \mathbb{N}$ be the corresponding permutation for $f + g : \mathbb{N} \to \mathbb{K}$, which is defined by pointwise addition.

Subadditivity:

$$
\begin{aligned}
\|f + g\|_{\mathrm{DTC}} &= \sum_{k=1}^{\infty} \left(\sqrt{k} - \sqrt{k-1}\right) \left|(f+g)\left(\pi_{f+g}(k)\right)\right| \\
&= \sum_{k=1}^{\infty} \left(\sqrt{k} - \sqrt{k-1}\right) \left|f\left(\pi_{f+g}(k)\right) + g\left(\pi_{f+g}(k)\right)\right| \\
&\leq \sum_{k=1}^{\infty} \left(\sqrt{k} - \sqrt{k-1}\right) \left(\left|f\left(\pi_{f+g}(k)\right)\right| + \left|g\left(\pi_{f+g}(k)\right)\right|\right) \\
&\leq \sum_{k=1}^{\infty} \left(\sqrt{k} - \sqrt{k-1}\right) (|f(k)| + |g(k)|) \quad \text{via Lemma 4.6 + Lemma 4.7}
\end{aligned}
$$

i.e. changing the permutation away from the mon. decreasing one can only make the sum smaller.

Absolute homogeneity: Let $s \in \mathbb{K}$. We have $|sf(k)| = |s||f(k)|$ for all $k \in \mathbb{N}$. Thus:

$$
\begin{aligned}
\|sf\|_{\mathrm{DTC}} &= \sum_{k=1}^{\infty} \left(\sqrt{k} - \sqrt{k-1}\right) |sf(k)| \\
&= |s| \sum_{k=1}^{\infty} \left(\sqrt{k} - \sqrt{k-1}\right) |f(k)| = |s| \|f\|_{\mathrm{DTC}}
\end{aligned}
$$

Positive definitivity:

$$
\begin{aligned}
\|f\|_{\mathrm{DTC}} = 0 &= \sum_{k=1}^{\infty} \left(\sqrt{k} - \sqrt{k-1}\right) |f(k)| \\
\iff |f(k)| &= 0 \quad \forall k \in \mathbb{N} \\
\iff f &\equiv 0
\end{aligned}
$$

□