# Resolving matroid circuit exchanges

Ellen Εμίλια Άννα Zscheile[1]

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## 1. Introduction

This paper discusses a way to decompose matroid circuit exchanges. These aren't really used anywhere, but we establish a connection to "margin-size neutral" basis exchanges given e.g. via pairs of tuples, which are used in Neil White's conjecture (NWC) about pairs of tuples of matroid bases.

First, we have to introduce some general definitions:

*Definition 1.1*: An **independence system** is a pair $(E, \mathcal{F})$, where $E$ (ground set) is a **finite** set (at least in the cases we consider), $\mathcal{F} \subseteq 2^E$ (independent sets), and such that the following holds:

- $\varnothing \in \mathcal{F}$ and
- $B \subseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$.

Cardinality-maximal independent subsets are called bases. The set of all bases of $(E, \mathcal{F})$ is called a **basis system**. ■

*Definition 1.2*: A **matroid** is an independence system such that additionally, the following holds:

$$A, B \in \mathcal{F}, |B| = |A| + 1 \Rightarrow \exists v \in B \setminus A \text{ with } A \cup \{v\} \in F$$

All bases have equal cardinality. ■

*Definition 1.3*: The circuit system $\mathcal{C}$ (system of the minimally dependent sets) of a matroid has the following axioms:

- $\varnothing \notin \mathcal{C}$,
- $A, B \in \mathcal{C}, A \subseteq B \Rightarrow A = B$ and
- $A, B \in \mathcal{C}, A \neq B, e \in A \cap B \Rightarrow \exists! C \in \mathcal{C} : C \subseteq (A \cup B) \setminus \{e\}$

The last property is matroid-specific; whereas the first two properties hold for all anti-chains. ■

For the rest of this paper, we consider just a fixed matroid $M$ with basis system $\mathcal{B}$ and circuit system $\mathcal{C}$.

**Theorem 1.4** (Symmetric exchange axiom (Proposition 4 in [1])): [For] bases $X, Y \in \mathcal{B}$, [...] for each $x \in X$, there exists $y \in Y$ such that $(X \setminus \{x\}) \cup \{y\}$ and $(Y \setminus \{y\}) \cup \{x\}$ are both bases. We say $x$ can be exchanged with $y$.

[1] https://orcid.org/0009-0005-9419-8568
RWTH Aachen University, Aachen, Germany

*Definition 1.5*: Given a field $k$ we consider the following morphism of polynomial rings:

$$\varphi_M : k[y_b]_{b \in \mathcal{B}} \to k[x_e]_{e \in E},$$
$$y_b \mapsto \prod_{e \in b} x_e \qquad \forall b \in \mathcal{B}$$

The definition above only defines the $\varphi_M$ for each basis, it should be extended linearly and multiplicatively to cover the entire domain. The **toric ideal** $I_M$ of a matroid $M$ is defined to be the kernel of $\varphi_M$. ■

According to [2] (and many other sources), the weak version of Neil White's conjecture states the following:

*Conjecture 1.6*: The toric ideal of $M$ is generated by quadratics. ■

In contrast, in this paper we manage to prove the following:

**Corollary 1.7**: The toric ideal of $M$ is generated by cubics and quadratics. ■

*Proof*: By Example 2.9 we can derive circuit deltas from instances of the NWC. Every element of the toric ideal of $M$ thus corresponds to a circuit delta. By Theorem 2.11 we can then find a linear combination of generators from Definition 2.5 which are equal to the circuit delta under consideration. Symmetric exchanges that stay inside the same circuit[2] correspond to quadratics.

The triangle exchanges from Definition 2.5 correspond to cubics (by removing the one appropriate element from the circuit, such that the entire exchange is contained in the kernel of $\varphi_M$ —see also example Example 2.9—, and extending that independent set to a basis), and the circuit exchanges similarly correspond to quadratics. Note that the property of triangle exchanges that they might extend the degree of the polynomial (by introducing a basis which wasn't in the polynomial before, due to the construction of the third circuit) doesn't concern the generators of ideals. □

**Theorem 1.8**: Every matroid has the following property:

$$\forall C_1, C_2 \in \mathcal{C}, e_1 \in C_1 \setminus C_2, e_2 \in C_2 \setminus C_1, C_1 \cap C_2 \neq \varnothing :$$
$$\exists C_3 \in \mathcal{C} : \{e_1, e_2\} \subseteq C_3$$

*Proof*: Let $\gamma_M$ be the equivalence relation[3] given by:

$$(e, f) \in \gamma_M \Leftrightarrow (e = f \vee \exists C \in \mathcal{C} : \{e, f\} \in C)$$

Let $C_1, C_2 \in \mathcal{C}$ and let $e_1 \in C_1 \setminus C_2$ and $e_2 \in C_2 \setminus C_1, e_3 \in C_1 \cap C_2$. Then we have that:

$$\begin{aligned} (e_1, e_3) &\in \gamma_M \quad \text{via } C_1 \\ \text{and} \quad (e_3, e_2) &\in \gamma_M \quad \text{via } C_2 \\ \Rightarrow (e_1, e_2) &\in \gamma_M \quad \text{by transitivity} \\ \Rightarrow \exists C_3 \in \mathcal{C} : \{e_1, e_2\} &\in C_3 \quad \text{because } e_1 \neq e_2 \end{aligned}$$

□

[2] i.e. those dropped by inspecting the circuit delta produced by Example 2.9 instead of the original NWC instance

[3] TODO: Cite something as to why this is relation is transitive. See e.g. https://iuuk.mff.cuni.cz/~pangrac/vyuka/matroids/matroid-ch5.pdf

# 2. Circuit deltas

*Definition 2.1*: We define the set of pointed circuits as $\mathcal{D} := \{(K, e) : e \in K \in \mathcal{C}\}$. ■

The main focus of this paper is the following construct:

*Definition 2.2*: We call $D : \mathcal{D} \to \mathbb{Z}$ a circuit delta for a matroid $M$ iff all of the following holds:
- $\forall K \in \mathcal{C} : 0 = \sum_{e \in K} D(K)(e)$, meaning each circuit is "balanced",
- $\forall e \in E : 0 = \sum_{K \in \mathcal{C} \text{ with } e \in K} D(K)(e)$, meaning each element is used as often in positive direction as in negative direction.

We call the set of all circuit deltas $\mathcal{D}'$. ■

*Definition 2.3*: Functions on pointed circuits $\mathbb{Z}^{\mathcal{D}}$ form a commutative group by inheriting the group operation from $(\mathbb{Z}, +)$, and also a $\mathbb{Z}$-module: Let $D, E : \mathcal{D} \to \mathbb{Z}$.
- the neutral function is $\hat{0}$ (usually also just called 0 when unambiguous),
- the inverse function is $-D$ for $D$,
- the sum of two functions $D$ and $E$ is $D + E$, and
- the $n \in \mathbb{Z}$ multiple of a function $D$ is $n \cdot D$.

They're defined by:

$$\begin{aligned}
\hat{0} &:= (K, e) \mapsto 0 \\
(-D) &:= (K, e) \mapsto -D(K, e) \\
(D + E) &:= (K, e) \mapsto D(K, e) + E(K, e) \\
(n \cdot D) &:= (K, e) \mapsto n \cdot D(K, e) \qquad \forall n \in \mathbb{Z}
\end{aligned}$$

■

**Lemma 2.4**: Circuit deltas $\mathcal{D}'$ form a $\mathbb{Z}$-submodule of $\mathbb{Z}^{\mathcal{D}}$.

*Proof*: The requirements from Definition 2.2 are trivially fulfilled for $\hat{0}$, $(-D)$, $(D + E)$, $(n \cdot D)$ from the Definition, assuming that one started with circuit deltas in the first place. □

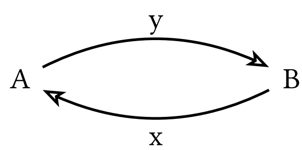
And we consider the following two generators for circuit deltas:

*Definition 2.5*: We consider the smallest $\mathbb{Z}$-submodule $\mathcal{D}'_{23}$ of circuit deltas generated[4] by the following:

**Circuit exchange**

$A, B \in \mathcal{C}, A \neq B, x, y \in A \cap B, x \neq y \Rightarrow$

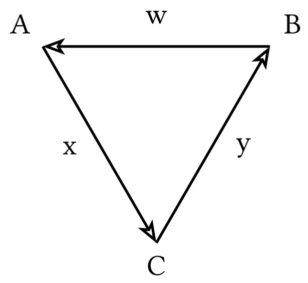
$$D_{2,A,B,x,y} \in \mathcal{D}' : (K, e) \mapsto \begin{cases} 1 & \text{for } (K, e) = (A, x), (B, y) \\ -1 & \text{for } (K, e) = (B, x), (A, y) \\ 0 & \text{otherwise} \end{cases}$$




[4]Note that they have a kind of symmetry such that the negated generator is also induced by a very similar generator of the same kind

**Triangle exchanges**

$$A, B \in \mathcal{C}, A \neq B, w \in A \cap B \Rightarrow$$
$$(\exists C \in \mathcal{C} : C \subseteq A \cup B)\text{: } \forall x \in A \cap C, y \in B \cap C\text{:}$$
$$D_{3,A,B,x,y,w} \in \mathcal{D}' : (K, e) \mapsto \begin{cases} 1 & \text{for } \ (K, e) = (A, w), (B, y), (C, x) \\ -1 & \text{for } \ (K, e) = (A, x), (B, w), (C, y) \\ 0 & \text{otherwise} \end{cases}$$




■

We want to inspect how or if these two generators already generate all circuit deltas for a given matroid $M$.

**Lemma 2.6**: For a triangle generator $D_{3,A,B,x,y,w}$ that isn't contained in the space spanned by circuit exchange generators, all of the following holds:

$$w \in (A \cap B) \setminus C; \quad x \in (A \cap C) \setminus B; \quad y \in (B \cap C) \setminus A$$

*Proof*: We prove the proposition about $x$ by contradiction. Let $x \in A \cap B \cap C$. Then $D_{2,A,B,x,w}$ is in $\mathcal{D}'_{23}$. Thus the following holds:

$$D_{3,A,B,x,y,w} = D_{2,C,B,x,y} + D_{2,A,B,w,x}$$

The proposition about $w, y$ follows analogously. □

**Lemma 2.7**: For two triangle generators $D_{3,A,B,x,y,w}$ and $D_{3,A,B,v,y,w}$ contained in $\mathcal{D}'_{23}$ with $x \neq v$, $x, v \in A \cap C$, then a circuit exchange can be used to transform them into each other.

*Proof*: $D_{3,A,B,x,y,w} - D_{3,A,B,v,y,w} = D_{2,A,C,v,x}$ □

**Corollary 2.8**: For each $A, B, C \in \mathcal{C}, A \neq B \neq C$, we only have to consider a single tuple $(x, y, w) \in ((A \cap C) \setminus B, (B \cap C) \setminus A, (A \cap B) \setminus C)$ yielding the triangle generator $D_{3,A,B,x,y,w}$, as all other triangle generators for the given $A, B$ can be produced by our singled-out triangle generator and the circuit exchange generators. ■

But first, we should take a look at how one encounters such "circuit deltas" in the wild.

*Example 2.9*: Consider a circular basis exchange given by $((B_i, e_i))_{i \in [t]} \subseteq \mathcal{B} \times E$ with $t \in \mathbb{N}, t \geq 2$, such that the following holds:

$$\forall i \in [t] : e_i \in B_{i-1} \setminus B_i, ((B_i \setminus \{e_{i+1}\}) \cup \{e_i\}) \in \mathcal{B}$$

Then we derive a similar "circuit exchange" $(C_i)_{i \in [t]}$:

$$\forall i \in [t] : \exists! C_i \in \mathcal{C} : \{e_i, e_{i+1}\} \subseteq C_i \subseteq (B_i \cup \{e_i\})$$

thanks to the matroid property that adding a single element to an independent set (e.g. basis) can lead to it containing at most one circuit (which is exploited e.g. in Edward's matroid intersection algorithm). We then "sum that up" into a circuit delta $D : \mathcal{D} \to \mathbb{Z}$, given by:

$$D(K,e) := \left|\{i \in [t] : K = C_i \wedge e = e_i\}\right| - \left|\{i \in [t] : K = C_i \wedge e = e_{i+1}\}\right|$$

This links circuit deltas to the results of my Bachelor thesis [3], in particular to the "directed closed trail extraction", which implicitly yields such a circular basis exchange. Circuit deltas are particularly interesting because they forget (as a forgetful functor) all "exchanges" / transfers of an element $e \in E \cap K$ that are contained in the same circuit $K \in \mathcal{C}$, which are easily "resolved" (can be reduced —to smaller circular basis exchanges— via symmetric exchanges that all use the same circuit), but annoying to have to deal with explicitly, given that they don't reveal any interesting structure about instances of the Neil White conjecture and are thus deemed relatively uninteresting in this context.

In particular, the generators defined in Definition 2.5 should be easier[5] to reduce to a sequence of symmetric exchanges (see Theorem 1.4) in the matroid. In fact, the generators $D_{2,A,B,x,y}$ are in direct correspondence to them. The generators $D_{3,A,B,x,y,w}$ are unfortunately necessary to be able to deal with parallel elements in the matroid, and maybe some similar "structures" in matroids which I couldn't pin down in a satifying manner, yet. ■

*Definition 2.10*: We consider the $\mathbb{R}$-vector space $K'$ (a sub-vector space of $\mathbb{R}^{\mathcal{D}}$) spanned by the following basis vectors:
- those constant on circuits: for each $K \in \mathcal{C}$, $\hat{K} : \mathcal{D} \to \mathbb{R}, (L, \bullet) \mapsto \begin{cases} 1 & \text{if } K=L \\ 0 & \text{otherwise} \end{cases}$,
- those constant on points of $M$: for each $e \in E$, $\hat{e} : \mathcal{D} \to \mathbb{R}, (\bullet, f) \mapsto \begin{cases} 1 & \text{if } e=f \\ 0 & \text{otherwise} \end{cases}$,

and call this the vector space of partially constant functions.[6] ■

Note that the above definition strictly speaking provides too many basis vectors. There are $\geq 2$ ways to generate the function constant on the entirety of $\mathcal{D}$: the sum of all functions constant on circuits, and the sum of all functions constant on points of $M$. If there are any loops in the matroid, these also lead to an overlap of partially constant functions.

**Theorem 2.11**: The $\mathbb{Z}$-module $\mathbb{Z}^{\mathcal{D}}$ is generated by the following generators:
- the restriction of the vector space of partially constant functions to the module $= K' \cap \mathbb{Z}^{\mathcal{D}}$,
- and the generators of $\mathcal{D}'_{23}$ introduced above.

*Proof*: Let $D : \mathcal{D} \to \mathbb{Z}$ be arbitrary. If $D$ is not a circuit delta, turn it into one by adding suitable $\mathbb{Z}$-multiples of the partially constant functions.

We prove the rest by downward induction over the "weight" of $D$, defined by:

$$\text{weight}(D) := \sum_{i \in \mathcal{D}} |D(i)|$$

If $D = \hat{0}$ (base case), we are done. Otherwise, we want to construct another circuit delta $D_+$ from $D$ with smaller weight, that only differs from $D$ by the addition/subtraction of finitely many generators.

Let $e \in E$ be arbitrary such that there exist $C_1, C_2 \in \mathcal{C}$ with $D(C_1, e) < 0$ and $D(C_2, e) > 0$ These must exist due to $D$ being a circuit delta, particularly $\sum_{e \in K \in \mathcal{C}} D(K, e) = 0$. It also follows that neither $C_1$ nor $C_2$ are loops.

[5]than attempting to solve the problem in full generality all at once

[6]This is necessary because just using those given basis functions as generators for a $\mathbb{Z}$-module doesn't lead to all the not-{"circuit deltas" / "margin size neutral functions"} parts of $\mathbb{Z}^{\mathcal{D}}$ being handled.

Let $f \in C_1, g \in C_2$ be arbitrary such that $D(C_1, f) > 0$ and $D(C_2, g) < 0$.

If $f = g$ or $f \in C_2$, then set $D_+ := D + D_{2,C_1,C_2,e,f}$.

Otherwise, if $g \in C_1$, then set $D_+ := D + D_{2,C_1,C_2,e,g}$.

Otherwise, there exists $C_3 \in \mathcal{C}$ with $\{f, g\} \subseteq C_3 \subseteq C_1 \cup C_2$ due to Theorem 1.8; then set $D_+ := D + D_{3,C_1,C_2,f,g,e}$.

And to finish the induction, we apply the induction hypothesis to $D_+$, with $\text{weight}(D_+) < \text{weight}(D)$. □

*Remark 2.12*: There exists a generalization of the above theorem to arbitrary anti-chains $\mathcal{C}$ instead of just those given by circuit systems of matroids, but it requires a generalization of $D_3$ (triangle) generators, given that the third circuit / anti-chain element no longer has to be unique, or exist at all. ■

*Conjecture 2.13*: In Definition 2.5, we can restrict the $D_3$ (triangle) generator such that it requires $\{w\} = A \cap B$ instead of $w \in A \cap B$, without that affecting the theorem above. ■

# 3. Notation

$\wedge$ logical and

$\vee$ logical or

$\oplus$ logical exclusive or

$\cap$ set intersection

$\cup$ set union

$\uplus$ disjoint set union

$\triangle$ symmetric set difference $X \triangle Y := (X \setminus Y) \cup (Y \setminus X)$

$\subseteq$ subset (or equal) of another set, i.e. $X \subseteq Y \Leftrightarrow X \cap Y = X$

$[t]$ natural numbers up to $t : \mathbb{N}_0$, defined as $[t] := \{1, ..., t\}$; $i + 1$ and $i - 1$ are defined to loop around $t, 1$.

$2^X$ for a given set $X$, $2^X$ denotes the powerset of $X$

$\binom{X}{k}$ for a given set $X$ and $k \in \mathbb{N}_0$, $\binom{X}{k}$ denotes the set of all $k$-element subsets of $X$

$\binom{n}{k}$ for $n, k \in \mathbb{N}_0$, $\binom{n}{k} := \left|\binom{[n]}{k}\right| = \frac{n!}{k!(n-k)!}$

$\exists$ object exists (e.g. in a set)

$\forall$ for all objects (e.g. in a set)

**flat of a matroid** Every inclusion-maximal dependent subset of the ground set is called a flat

**loop of a matroid** Every one-element circuit of a matroid is called a loop (usually used interchangably with saying the element contained in the loop circuit is a loop).

**points of a matroid** For a matroid $M = (E, \mathcal{F})$, the elements of $M$ are called its *points*.

**w.l.o.g.** without loss of generality

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