CIRCUIT EXCHANGES AND THE TORIC IDEAL OF A MATROID

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Abstract. This article proves that the toric ideal of a matroid is generated in degree at most 3, which is relevant in regards to the weak variant of Neil White's conjecture about matroid basis tuples and the associated toric ideal.

1. Introduction

Let M be a fixed matroid with ground set E, set of bases $\mathcal{B}$, set of circuits $\mathcal{C}$ and rank function r.

Definition 1.1. The **connection relation** γ_M of M is defined [4, Chapter 4] by:

$$(e, f) \in \gamma_M \iff (e = f \vee \exists C \in \mathcal{C} : \{e, f\} \in C)$$

Proposition 1.2 ([4, Proposition 4.1.2.])**.** *The* connection relation γ_M *is an equivalence relation.*

Definition 1.3. Define the function $\hat{\mathcal{C}} : (B : \mathcal{B}) \to (E \setminus B) \to \mathcal{C}$, by requiring for each $B \in \mathcal{B}, e \in E \setminus B$, that $\hat{\mathcal{C}}(B)(e)$ is the unique circuit that gets introduced into B when adding e to it.

Definition 1.4. Given a field $\mathbb{K}$ we consider the following homomorphism of polynomial rings:

$$\varphi_M : \mathbb{K}[y_b]_{b \in \mathcal{B}} \to \mathbb{K}[x_e]_{e \in E},$$
$$y_b \mapsto \prod_{e \in b} x_e, \quad \forall b \in \mathcal{B}$$

The definition above only defines the φ_M for each basis, it should be extended linearly and multiplicatively to cover the entire domain. The **toric ideal** of a matroid M is defined [1] to be the kernel of φ_M.

Let $\xrightarrow{(k,e)}$ denote an operation [3] acting on an independent set $F \in \mathcal{F}$ that removes $k \in F$ and adds $e \in E \setminus F$; and if $k = e$, it does nothing.

According to [1], the weak version of Neil White's conjecture states the following:

Conjecture 1.5. *The toric ideal of M is generated by quadratics.*

...and the same paper proves the following, which is, as far as we know, the state of the art:

Theorem 1.6. *The toric ideal of M is generated in degree at most* $(r(E) + 3)!$

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2. Circuit exchanges and the toric ideal of a matroid

In contrast, in this paper we manage to prove the following:

Theorem 2.1. *The toric ideal of M is generated in degree at most 3.*

But first, we need a few utilities:

Lemma 2.2. *Let $p \in \mathbb{K}[y_b]_{b\in\mathcal{B}}$ and $B \in \mathcal{B}$. Then $y_B p \in \ker\varphi_M$ holds if and only if $p \in \ker\varphi_M$. And $p\, y_B \in \ker\varphi_M$ holds if and only if $p \in \ker\varphi_M$.*

Proof.

$$0 = \varphi_M(y_B p) \overset{\varphi_M \text{ hom.}}{=} \underbrace{\varphi_M(y_B)}_{\neq 0}\varphi_M(p) \quad\iff\quad 0 = \varphi_M(y_B)^{-1}\varphi_M(y_B p) = \varphi_M(p)$$

And the other equivalence holds analogously. □

Lemma 2.3. *Let $\mathfrak{a}$ be an ideal in a commutative polynomial ring $\mathbb{K}[z_i]_{i\in I}$ with coordinate index set I, such that no non-zero generator of $\mathfrak{a}$ has a non-constant monomial that can be factored out.*

Let $j \in I$. Then $z_j p \in \mathfrak{a}$ holds if and only if $p \in \mathfrak{a}$ (and the same holds for $p\, z_j$ instead of $z_j p$).

Proof. "$\Leftarrow$" is clear because $\mathfrak{a}$ is an ideal.

For the other direction, we proceed by contradiction and assume $z_j p \in \mathfrak{a}$ holds, but $p \in \mathfrak{a}$ doesn't, and that p has minimal degree in this regard.

Then there exist some $z_j q \in \mathfrak{a}$ and $q, r \in \mathbb{K}[z_i]_{i\in I}$ such that $qr = p$, with r of maximal degree. Note in particular that $z_j q \in \mathfrak{a}$ follows from the fact that $z_j p \in \mathfrak{a}$, and because omitting/"dividing by" the z_j leaves the ideal, there needs to exist such a polynomial q (which has at most the degree of p).

We have that $q \notin \mathfrak{a}$ because $p = qr \notin \mathfrak{a}$. Thus it is clear that r is a constant $\in \mathbb{K}$ because p has minimal degree. W.l.o.g. assume $r = 1$, and therefore get $q \notin \mathfrak{a}$. Thus $p = q$, and $z_j p$ is a generator of $\mathfrak{a}$.

Additionally, we know that $p \neq 0$ because $0 \in \mathfrak{a}$. So we have found a non-zero generator $z_j p$ of $\mathfrak{a}$ that has a non-constant monomial z_j that can be factored out, which contradicts our asusmptions. Therefore p must be in $\mathfrak{a}$. □

Definition 2.4. Let $p \in \ker\varphi_M$ be arbitrary. W.l.o.g., assume $p = y_{B_1}y_{B_2}\cdots y_{B_t} - y_{D_1}y_{D_2}\cdots y_{D_t}$ with $t \in \mathbb{N}_0$ and $B_1, \ldots, B_t, D_1, \ldots, D_t \in \mathcal{B}$.

The symmetric difference between bases has always even size. Define the weight of the difference between the two monic polynomials as follows:

$$\operatorname{weight}(p) := \frac{1}{2} \min_{\substack{\pi:\{1,\ldots,t\}\to\{1,\ldots,t\}\\ \pi \text{ bijective}}} \sum_{i=1}^{t} \left|B_i \triangle D_{\pi(i)}\right|$$

For comparison, consider the overlap function used in [2, Section 2], which uses $|B_i \cap D_{\pi(i)}|$ instead. We can't easily reuse exactly that function here because we might need to insert additional bases into the polynomial.

Proof of theorem 2.1. Let $p \in \ker\varphi_M$ be arbitrary. W.l.o.g., assume

$$y_{B_1}y_{B_2}\cdots y_{B_t} - y_{D_1}y_{D_2}\cdots y_{D_t} := p \text{ with } t \in \mathbb{N}_0 \text{ and}$$
$$B_1, \ldots, B_t, D_1, \ldots, D_t \in \mathcal{B}.$$

W.l.o.g., assume the identity realizes the minimum in weight(p) and $B_i \neq D_i$ for all $i = 1, \ldots, t$ (otherwise divide by the common factor $y_{B_i} = y_{D_i}$ using lemma 2.2).

We proceed by performing induction on weight(p). The base cases are $t = 0, 2, 3$ and weight(p) $= 0, 2, 3$ because those are the generators whose presence we assume + the null polynomial.

Analogouos to [2, Proof of Theorem 2], suppose the assertion holds for all binomials $p' \in \ker \varphi_M$ with $\text{weight}(p') < \text{weight}(p)$. Let $t \geq 2, \text{weight}(p) > 3$. Let $i, j = 1, \ldots, t$ with $i \neq j$ and $e \in D_i \setminus B_i, f \in (B_i \setminus D_i) \cap (D_j \setminus B_j), g \in B_j \setminus D_j$, such that:

$$e, f \in \hat{\mathcal{C}}(B_i)(e) =: C_i$$
$$f, g \in \hat{\mathcal{C}}(B_j)(f) =: C_j$$

The existence of i, j, e, f, g follows from $p \in \ker \varphi_M$ and $t \geq 3$, and the existence of C_i, C_j follows from the matroid property that adding an element to a basis which wasn't included before introduces exactly one circuit.

Thus we have that $(e, f), (f, g) \in \gamma_M$, and by transitivity $(e, g) \in \gamma_M$. If $e \neq g$, then

$$\exists \tilde{C} \in \mathcal{C} : \quad e, g \in \tilde{C}$$

Choose $\tilde{B} \in \mathcal{B}$ with $g \notin \tilde{B}$ arbitrarily, and if $e \neq g$ additionally require $\tilde{C} \setminus \{g\} \subseteq \tilde{B}$, which exists because we can extend every independent set to a basis. Set:

$$F_i := B_i \cup \{e\} \setminus \{f\} \qquad (B_i \xrightarrow{(f,e)} F_i)$$
$$F_j := B_j \cup \{f\} \setminus \{g\} \qquad (B_j \xrightarrow{(g,f)} F_j)$$
$$\tilde{F} := \tilde{B} \cup \{g\} \setminus \{e\} \qquad (\tilde{B} \xrightarrow{(e,g)} \tilde{F})$$

Thus $y_{B_i} y_{B_j} y_{\tilde{B}} - y_{F_i} y_{F_j} y_{\tilde{F}} \in \ker \varphi_M$ by base case (if $e = g$, then $\tilde{B} = \tilde{F}$). Note that:

$$(1) \qquad |B_l \triangle D_l| = 2 + |F_l \triangle D_l|, \quad \forall l = i, j$$

Therefore we proceed in the next induction step with:

$$y_{\tilde{B}} \left(y_{B_1} y_{B_2} \cdots y_{B_t} - y_{D_1} y_{D_2} \cdots y_{D_t} \right) - \left(y_{B_i} y_{B_j} y_{\tilde{B}} - y_{F_i} y_{F_j} y_{\tilde{F}} \right) \prod_{l \in \{1, \ldots, t\} \setminus \{i,j\}} y_{B_l}$$
$$= y_{F_i} y_{F_j} y_{\tilde{F}} \prod_{l \in \{1, \ldots, t\} \setminus \{i,j\}} y_{B_l} \quad - y_{\tilde{B}} y_{D_1} y_{D_2} \cdots y_{D_t} \quad =: p'$$

We are allowed to do so because for the weight, the following holds, and we assumed that the induction assumption holds for all p' with $\text{weight}(p') < \text{weight}(p)$:

$$2 \,\text{weight}(p) - 2 \,\text{weight}(p')$$
$$\geq 2 \,\text{weight}(p) - |F_i \triangle D_i| - |F_j \triangle D_j| - \left| \tilde{F} \triangle \tilde{B} \right| - \sum_{l \in \{1, \ldots, t\} \setminus \{i,j\}} |B_l \triangle D_l|$$
$$= 4 - \underbrace{\left| \tilde{F} \triangle \tilde{B} \right|}_{\leq 2} \geq 2 \qquad \text{(due to (1))}$$

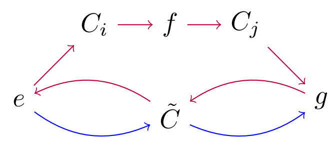
In summary:

$$\text{By construction,} \qquad y_{B_i} y_{B_j} y_{\tilde{B}} - y_{F_i} y_{F_j} y_{\tilde{F}} = p'' \in \ker \varphi_M$$
$$\text{By induction assumption,} \qquad y_{\tilde{B}} p - p'' \prod_{l \in \{1, \ldots, t\} \setminus \{i,j\}} y_{B_l} = p' \in \ker \varphi_M$$
$$\Longrightarrow \qquad y_{\tilde{B}} p = p' + p'' \prod_{l \in \{1, \ldots, t\} \setminus \{i,j\}} y_{B_l} \in \ker \varphi_M$$
$$\overset{\text{using lemma 2.2}}{\Longleftrightarrow} \qquad p \in \ker \varphi_M$$

$$\text{we decompose:} \qquad e \longrightarrow C_i \longrightarrow f \longrightarrow C_j \longrightarrow g$$

Presented visually,

into:



∎

Remark 2.5. For the case $t = 2$, we have that $\varnothing = \{1, \dots, t\} \setminus \{i, j\}$, thus the summary of the above proof looks like this if $e \neq g$ (apply triangle cyclic exchange):

$$\begin{aligned} \text{By construction,} \qquad & y_{B_i} y_{B_j} y_{\tilde{B}} - y_{F_i} y_{F_j} y_{\tilde{F}} = p'' \in \ker \varphi_M \\ \text{By induction assumption,} \qquad & p\, y_{\tilde{B}} - p'' = p' \in \ker \varphi_M \\ \implies \qquad & p\, y_{\tilde{B}} = p' + p'' \in \ker \varphi_M \\ \overset{\text{using lemma 2.2}}{\iff} \qquad & p \in \ker \varphi_M \end{aligned}$$

If $e = g$, we can use the following, equivalent (via lemma 2.2) summary instead (apply symmetric exchange):

$$\begin{aligned} \text{By construction,} \qquad & y_{B_i} y_{B_j} - y_{F_i} y_{F_j} = p'' \in \ker \varphi_M \\ \text{By induction assumption,} \qquad & p - p'' = p' \in \ker \varphi_M \\ \iff \qquad & p = p' + p'' \in \ker \varphi_M \end{aligned}$$

In particular, this conclusion ($p \in \ker \varphi_M$ is generated by triangle cyclic exchanges and symmetric exchanges) also holds when using the non-commutative polynomial ring $\mathbb{K}[[y_b]]_{b \in \mathcal{B}}$ instead of the commutative polynomial ring $\mathbb{K}[y_b]_{b \in \mathcal{B}}$, since we don't have to rely on permuting p anywhere.

Subsequently, we can generalize the theorem 2.1 further, by utilizing the following lemma derived from [2, Lemma 12]:

Lemma 2.6. *For any matroid M the following are equivalent:*

(1) *The toric ideal of M in the non-commutative case is generated in degree at most 3, by the generators of triangle cyclic exchanges and symmetric exchanges.*
(2) *The toric ideal of M in the commutative case is generated in degree at most 3, by the generators of triangle cyclic exchanges and symmetric exchanges.*

Proof. Implication (1) $\implies$ (2) is clear from the definition. To get the opposite implication it is enough to recall that any permutation is a composition of transpositions, and by remark 2.5, even the non-commutative representation of the transposition of $B_1, B_2 \in \mathcal{B}$ as $y_{B_1} y_{B_2} - y_{B_2} y_{B_1}$ is generated by triangle cyclic exchanges and symmetric exchanges. ∎

Corollary 2.7. *The toric ideal of M in the non-commutative case is generated in degree at most 3, by the generators for triangle cyclic exchanges and symmetric exchanges.*

Proof. This follows directly from combining theorem 2.1 with lemma 2.6. ∎

3. Reducing triangle circuit exchanges

In case we have additional overlap inside of a triangle (cyclic) exchange, we can reduce it further:

Let $B_1, B_2, B_3 \in \mathcal{B}$ and $e \in B_3 \setminus B_1$, $f \in B_1 \setminus B_2$, $g \in B_2 \setminus B_3$, such that

$$B_1 \cup \{e\} \setminus \{f\},\ B_2 \cup \{f\} \setminus \{g\},\ B_3 \cup \{g\} \setminus \{e\} \in \mathcal{B}$$

Then we have that:

$$f \in C_1 := \hat{\mathcal{C}}(B_1)(e), \quad g \in C_2 := \hat{\mathcal{C}}(B_2)(f), \quad e \in C_3 := \hat{\mathcal{C}}(B_3)(g)$$

Lemma 3.1. *If $\{e, f, g\} \cap C_1 \cap C_2 \cap C_3 \neq \emptyset$ holds, then this cyclic exchange is expressable by two symmetric exchanges.*

Proof. W.l.o.g. let $f \in C_1 \cap C_2 \cap C_3$. Exchange f, g between B_2 and B_3: $B_3 \cup \{g\} \setminus \{f\} \in \mathcal{B}$, and then exchange e, f between B_1 and $B_3 \cup \{g\} \setminus \{f\}$. □

Lemma 3.2. *If $k \in C_1 \cap C_2 \cap C_3 \neq \emptyset$, then this cyclic exchange is expressable by at most four symmetric exchanges, utilizing at most one additional dummy base.*

Proof. Let $k \notin \{e, f, g\}$ (otherwise use the previous lemma).

$$B_1' = B_1 \cup \{e\} \setminus \{f\}$$
$$B_1'' = B_1 \cup \{e\} \setminus \{k\}$$

Then we use the following exchanges:

$$\begin{array}{lllll}
B_1 \xrightarrow{(k,e)} B_1'' \xrightarrow{(f,k)} B_1' & & \\
B_2 & \xrightarrow{(k,f)} B_2'' \xrightarrow{(g,k)} B_2' & \\
B_3 & & \xrightarrow{(k,g)} B_3'' \\
B_3'' \xrightarrow{(e,k)} B_3' & &
\end{array}$$

□

Lemma 3.3. *If $\{e, g\} \in \mathcal{C}$, then this cyclic exchange is expressable by two symmetric exchanges.*

Proof. Use the following exchanges:

$$\begin{array}{lll}
B_1 & & \xrightarrow{(f,e)} B_1' \\
B_2 & \xrightarrow{(g,e)} B_2'' & \xrightarrow{(e,f)} B_2' \\
B_3 & \xrightarrow{(e,g)} B_3' &
\end{array}$$

□

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