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Justifying division in basis systems

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16. June 2026

This article investigates when division by / cancellation of monomials works in homogeneous toric ideals, specifically concerning toric ideals of matroids and their subideals generated by quadratic binomials, with applications to the Neil White conjecture.

1 Acknowledgements

I'd like to thank Prof. Ondřej Pangrác (Charles University, Prague), Prof. Mateusz Michałek (University of Konstanz) and Tali Auster (RWTH Aachen University) for their feedback and occasionally crucial insights which lead to the creation of this article. This article is a continuation of the work of my Bachelor thesis, which (the thesis) was advised by Christoph Helmberg and Kurt Klement Gottwald (University of Technology Chemnitz).

2 Introduction

Let E be a fixed *ground* set, whose elements are called *points*. Let $\mathcal{B} \subseteq \binom{E}{r}$ be a fixed set of *bases* with $r \in \mathbb{N}$ called the *rank*. Let $\mathbb{K}$ be a field with characteristic zero.

Definition 2.1. We consider the following homomorphism of polynomial rings:

$$\varphi_{\mathcal{B}} : \mathbb{K}[y_b]_{b \in \mathcal{B}} \to \mathbb{K}[x_e]_{e \in E},$$
$$y_b \mapsto \prod_{e \in b} x_e \quad \forall b \in \mathcal{B}$$

The definition above only defines the $\varphi_{\mathcal{B}}$ for each basis, it should be extended linearly and multiplicatively to cover the entire domain. The **toric ideal** of a matroid M with bases $\mathcal{B}$ is defined [1] to be the kernel of $\varphi_{\mathcal{B}}$. ■

2020 Mathematics Subject Classification. 05B35, 05C40, 13F65
Key words and phrases. toric ideals of matroids
* orcid.org/0009-0005-9419-8568

Definition 2.2.

$$\hat{L}_{\mathcal{B}} := \{y_a y_b : a, b \in \mathcal{B}\}; \qquad L_{\mathcal{B}} := \varphi_{\mathcal{B}}(\hat{L}_{\mathcal{B}})$$

We want to inspect the following ideal:

$$I_{\mathcal{B}} := \langle v - z \in \ker \varphi_{\mathcal{B}} : v, z \in \hat{L}_{\mathcal{B}} \rangle \subset \ker \varphi_{\mathcal{B}}$$

And we also need the ideals per *ground level*:

$$\psi_{\mathcal{B}} := (\varphi_{\mathcal{B}}|_{\hat{L}_{\mathcal{B}}})^{-1} : L_{\mathcal{B}} \to 2^{\hat{L}_{\mathcal{B}}}$$

$$I_{\mathcal{B};l} := \langle v - z : v, z \in \psi_{\mathcal{B}}(l) \rangle \subset I_{\mathcal{B}} \qquad \forall l \in L_{\mathcal{B}} \tag{1}$$

■

3 Division by monomials

The problem we actually want to tackle is that although we can divide polynomials that have y_b as a factor by y_b (for $b \in \mathcal{B}$), and stay inside (it is closed w.r.t. this property) the polynomial ring and $\ker \varphi_{\mathcal{B}}$, it isn't immediately clear if this also holds in the ideal $I_{\mathcal{B}}$, i.e. that this ideal is saturated w.r.t. the irrelevant ideal $\langle y_b : b \in \mathcal{B} \rangle$, thus this article is concerned with proving exactly that, which should suffice to solve what was still open to prove the Neil White conjecture [2]. First, we have to introduce a notion where we *canonicalize* / *center* around a single $z \in \psi_{\mathcal{B}}(l)$ for each $l \in L_{\mathcal{B}}$.

Lemma 3.1. *Let $R = \mathbb{K}[v_i]_{i \in I}$ be a polynomial ring with variables indexed by an arbitrary finite set I. Let $G \subset R$ be a set of monomials. Then the following ideals are equal:*

1. $\langle w - z : w, z \in G \rangle$
2. *For $z \in G$: $\langle w - z : w \in G, w \neq z \rangle$*

and if every pair of generators is relatively prime, then the latter are generated by a minimal set of generators.

Proof. Use the correspondence $w_1 - w_2 = (z - w_2) - (z - w_1)$ for each $w_1, w_2, z \in G$, meaning all of these ideals are all equal. □

Then we want to leverage the following:

Theorem 3.2. *Let $R = \mathbb{K}[v_i]_{i \in I}$ be a polynomial ring with variables indexed by an arbitrary finite set I. Let $G \subset R$ be a set of monomials such that all of them have the same degree, and every pair of them is relatively prime.*

Then $\mathfrak{a} := \langle w - z : w, z \in G \rangle$ is saturated w.r.t. the irrelevant ideal $\langle v_i \rangle_{i \in I}$.

Proof. Let $i \in I$. Then $v_i p \in \langle w - z : w, z \in G \rangle$ implies $p \in \langle w - z : w, z \in G \rangle$ because no generator of $\mathfrak{a}$ is divisible by v_i, and therefore every prefactor for a generator has to be divisible by v_i. □

Remark 3.3. Note that $\langle G\rangle$ is not saturated w.r.t. the irrelevant ideal because e.g. $1 \neq z = \prod_{i\in I_1} v_i \in G \implies z \in \langle G\rangle$, but for each $i \in I_1$ we have that $v_i^{-1}z = \prod_{j\in I_1, j\neq i} v_j \notin \langle G\rangle$. ■

Corollary 3.4. *Let $l \in L_{\mathcal{B}}$, $i \in \mathcal{B}$ and $p \in \mathbb{K}[y_b]_{b\in\mathcal{B}}$. Then $I_{\mathcal{B};l}$ is saturared and we have that $y_i p \in I_{\mathcal{B};l} \iff p \in I_{\mathcal{B};l}$.*

Proof. Use theorem 3.2 with $I \leftarrow \mathcal{B}$, $G \leftarrow \psi_{\mathcal{B}}(l)$, to get that $I_{\mathcal{B};l}$ is saturated. □

And thus we get the final result:

Theorem 3.5. *Let $i \in \mathcal{B}$ and $p \in \mathbb{K}[y_b]_{b\in\mathcal{B}}$. Then $y_i p \in I_{\mathcal{B}} \iff p \in I_{\mathcal{B}}$.*

Proof. ($\Leftarrow$) is clear, so only ($\Rightarrow$) remains to be shown. If $p = 0$, we are done. Otherwise, we can then represent $y_i p \in I_{\mathcal{B}}$ as follows, such that the amount of non-zero $p^{(l)}$ is minimal:

$$y_i p = \sum_{l\in L_{\mathcal{B}}} \underbrace{y_i p^{(l)}}_{\in I_{\mathcal{B};l}}$$

We can apply corollary 3.4, from which we get $p^{(l)} \in I_{\mathcal{B};l}$, and $p = \sum_{l\in L_{\mathcal{B}}} p^{(l)} \in I_{\mathcal{B}}$ □

References

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