Graphs and transitive closures

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Contents

1. Introduction

This paper proves Theorem 3.8 about generated graphs, the kind one encounters when attempting to deal with the Neil White conjecture(s), and describes how disconnecting a generated graph by modifications of the original one also disconnects the original one. Subsequently, it also concludes that for a bunch of variants of the Neil White conjecture, it suffices to prove that the NWC holds for multiple symmetric exchanges for all $t \geq 2$, and the variant follows via Theorem 4.3.1.

1.1. Acknowledgements

I am grateful to Locria Cyber ("iacore"), Tara Geißler ("42triangles")[2] and Kurt Klement Gottwald[3] for proofreading early revisions of this document.

2. Definitions

Definition 2.1: Let V be a set. Let $E \subseteq \binom{V}{2}$. Then (V, E) is called a **graph**, and E are called the **edges** of (V, E). ■

Definition 2.2: Let (V, E) be a graph. Let $A, B \in V$. Then we define $P(E, A, B)$ as the set of all **paths** between A and B (meaning all connections between A and B such that per path, no vertex $\in V$ gets encountered twice). ■

Note that $P(E, A, B) = P(E, B, A)$.

Proposition 2.3: Let $(\mathcal{F}, E)$ be a graph. If $\{a, b\} \in E$, then $\{\{a, b\}\} \in P(E, a, b)$.

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Definition 2.4: Let $\mathcal{F}$ be a set. Let $E \subseteq \binom{\mathcal{F}^2}{2}$. We call E a **generating edge set** iff for each $\{(x_1, x_2), (y_1, y_2)\} \in E$, we have that all of the following hold:

$$(x_1 \neq y_1) \quad \wedge \quad (x_2 \neq y_2) \quad \wedge \quad \{(x_2, x_1), (y_2, y_1)\} \in E$$

■

The inequality restrictions are necessary to prevent scenarios where $x_1 = y_1 \wedge x_2 \neq y_2$ and such.

3. Liftings and transitive closure properties

Let $\mathcal{F}$ be a set. Let $E \subseteq \binom{\mathcal{F}^2}{2}$ be a *generating edge set*. Let $t \in \mathbb{N}, t \geq 2$.

Remark 3.1: Given a reflexive, symmetric relation $R \subseteq \mathcal{F}^2$ (which corresponds to an undirected graph, lets call it $G = (\mathcal{F}, E)$):

$$E := \{\{x, y\} : x, y \in \mathcal{F}, x \neq y, xRy\} \subseteq \binom{\mathcal{F}}{2}$$

the equivalence classes of the resulting equivalence relation after taking the transitive closure of R are in bijective correspondence with the graph components of the undirected graph G, see also [1]. ■

Definition 3.2 induced graph of tuples: Define $\mathrm{mktup}_t\, E \subseteq \binom{\mathcal{F}^t}{2}$, by stating that $\{x, y\} \in \mathrm{mktup}_t\, E$ if and only if $\exists \{m, n\} \in \binom{[t]}{2}$ such that the following holds:

$$\{(x_m, x_n), (y_m, y_n)\} \in E \quad \wedge \quad (\forall i \in [t] \setminus \{m, n\} : x_i = y_i)$$

■

Proposition 3.3: $\mathrm{mktup}_t\, E$ is equal to $\uplus_{e \in E} \mathrm{mktup}_t \{e\}$.

Lemma 3.4: $\mathrm{mktup}_2\, E$ is equal to E.

Proof: Because $\binom{[2]}{2} = \{\{1, 2\}\}$ holds, we have $\{m, n\} = \{1, 2\}$. Thus, $\{x, y\} \in \mathrm{mktup}_2\, E$ if and only if $\{(x_1, x_2), (y_1, y_2)\} \in E$ which is well-defined due to the symmetry requirement on generating edge sets. □

Definition 3.5 unlift: Let $\{c, d\} \in \mathrm{mktup}_t\, E$. Then define $\{a, b\} := \mathrm{unlift}_{\mathcal{F}} \{c, d\} \in E$ such that $\exists! \{m, n\} \in \binom{[t]}{2}$ with

$$\{a, b\} := \{(c_m, c_n), (d_m, d_n)\} \in E \qquad \text{and} \qquad \forall i \in [t] \setminus \{m, n\} : c_i = d_i.$$

■

Proof: That $\{a, b\}$ is unique and well-defined follows from the definition of generating edge sets and $\mathrm{mktup}_t \{\{c, d\}\}$. □

Lemma 3.6 (lifting of paths): We have that $\forall \{c, d\} \in \mathrm{mktup}_t\, E$: Set $\{a, b\} := \mathrm{unlift}_{\mathcal{F}} \{c, d\}$. For every *generating edge set* $S \subseteq E$ and $q \in P(S, a, b)$, there exists a corresponding entry $p \in P(\mathrm{mktup}_t\, S, c, d)$.

Proof: Assume we have t, a, b, c, d as above. There $\exists!\{m, n\} \in \binom{[t]}{2}$ given implicitly in the definition of unlift.

Let $S \subseteq E$ be a generating edge set and $q \in P(S, a, b)$. Then we can construct the corresponding $p \in P(\text{mktup}_t\, S, c, d)$ as follows: $\{e, f\} \in p$ if and only if both:

$\{(e_m, e_n), (f_m, f_n)\} \in q$ and $\forall i \in [t] \setminus \{m, n\} : c_i = e_i = f_i = d_i$ hold.

That this is again a valid path follows from the definition of mktup_t. □

Corollary 3.7: $P(\text{mktup}_t\, S, c, d) = \emptyset \Rightarrow P(S, a, b) = \emptyset$ ■

In the theorem below, generating edge set $S \subseteq E$ is to be interpreted as something like a "cut-set", but having the purpose of cutting $\text{mktup}_t\, E$ (and it follows that it is actually also a cut-set for E itself).

Theorem 3.8: Let $A, B \in \mathcal{F}^t$.

Let $S \subseteq E$ be a *generating edge set* such that $\forall p \in P(\text{mktup}_t\, E, A, B) : p \cap (\text{mktup}_t\, S) \neq \emptyset$.

Then it follows that $\forall p \in P(\text{mktup}_t\, E, A, B) : \exists\{a, b\} \in S : \exists\{c, d\} \in p \cap (\text{mktup}_t\{\{a, b\}\})$:

$$P\left(\underset{t}{\text{mktup}}(E \setminus S), c, d\right) \overset{4}{=} \emptyset \wedge P(E, a, b) \neq \emptyset \wedge P(E \setminus S, a, b) = \emptyset$$

Proof: Let $t \in \mathbb{N}, t \geq 2$. Let $A, B \in \mathcal{F}^t$. Let $S \subseteq E$ as above.

Let $p \in P(\text{mktup}_t\, E, A, B)$ be arbitrary. Then there exists an $\{c, d\} \in p \cap (\text{mktup}_t\, S)$. Thus there exists a corresponding $\{a, b\} \in S$ such that $\{c, d\} \in p \cap (\text{mktup}_t\{\{a, b\}\})$.

Now it remains to prove that we can choose a, b, c, d such that the following holds:

$$P\left(\underset{t}{\text{mktup}}(E \setminus S), c, d\right) \overset{4}{=} \emptyset$$

The rest follows from that:

$$\begin{aligned} P(E, a, b) \neq \emptyset \quad & \text{follows from Prop. 2.3,} \\ P(E \setminus S, a, b) = \emptyset \; & \text{follows from Corollary 3.7} \end{aligned}$$

If there wouldn't exist any $\{c, d\} \in p \cap \text{mktup}_t\{\{a, b\}\}$ such that $P(\text{mktup}_t(E \setminus S), c, d) = \emptyset$ holds, then there would exist a path between all those c and d's. But then there would exist some path $p' \in P(\text{mktup}_t\, E, A, B)$ with $p' \cap (\text{mktup}_t\, S) = \emptyset$ because we could replace each $\{c, d\}$ edge in p with an arbitrary path from $P(\text{mktup}_t(E \setminus S), c, d)$, yielding a walk that can be contracted to a path p'. This is a contradiction to the assumption above. □

The meaning of Theorem 3.8 is that when we remove the edges in S such that every path between A and B gets interrupted, then it follows that this corresponds to an interruption of paths in the generating graph $(\mathcal{F}^2, E)$, meaning that cuts in "higher" ($t \geq 2$) graphs $(\mathcal{F}^t, \text{mktup}_t\, E)$ lead to cuts in the generating graph.

[4]meaning that $\{a, b\}$ is critical

4. Application: The Neil White Conjecture

4.1. Setting / Context

This section was mostly extracted from the thesis [2].

The following definitions were extracted from lecture notes about "Introduction to discrete mathematics", a lecture held by Christoph Helmberg, partially translated from [3] and influenced by [4].

Definition 4.1.1: An **independence system** is a pair $(E, \mathcal{F})$, where E (ground set) is a **finite** set (at least in the cases we consider), $\mathcal{F} \subseteq 2^E$ (independent sets), and such that the following holds:

- $\emptyset \in \mathcal{F}$ and
- $B \subseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$.

Cardinality-maximal independent subsets are called bases. The set of all bases of $(E, \mathcal{F})$ is called a **basis system**. ■

Definition 4.1.2: A **matroid** is an independence system such that additionally, the following holds:

$$A, B \in \mathcal{F}, |B| = |A| + 1 \Rightarrow \exists v \in B \setminus A \text{ with } A \cup \{v\} \in F$$

All bases have equal cardinality. ■

4.1.1. Matroid base exchange properties

Theorem 4.1.1.1 (Symmetric exchange axiom (Proposition 4 in [5])): If $\mathcal{B}$ is a basis system [of a matroid] with bases $X, Y \in \mathcal{B}$, then for each $x \in X$, there exists $y \in Y$ such that $(X \setminus \{x\}) \cup \{y\}$ and $(Y \setminus \{y\}) \cup \{x\}$ are both bases. We say x can be exchanged with y.

Lemma 4.1.1.2 (Multiple symmetric exchange [6]): Let X and Y be bases of a [matroid][5] G. Then for any subset $A \subseteq X$, there exists a subset $B \subseteq Y$ with the property that $(X \setminus A) \cup B$ and $(Y \setminus B) \cup A$ are both bases of B.

4.1.2. The Neil White conjecture(s)

This conjecture was originally introduced in [7], and a good summary can be found in [8] by Michał Lasoń. These aren't really compact enough to be easy to quote directly here. We paraphrase them here instead.

Definition 4.1.2.1 Compatibility: Let E be a finite set (called the "ground set"). We call two multisets $X, Y : 2^E \to \mathbb{N}_0$ in 2^E **compatible** (denoted by $X \underset{0}{\sim} Y$) if their multiset union is equal, i.e.

$$\sum_{e \in V \subseteq E} X(V) = \sum_{e \in V \subseteq E} Y(V) \quad \forall e \in E$$

and they have an equal amount of members per size, i.e.

[5]The source calls "matroid" "combinatorial geometry"

$$\sum_{V \in \binom{E}{n}} X(V) = \sum_{V \in \binom{E}{n}} Y(V) \quad \forall n \in \mathbb{N} \tag{1}$$

Let $t \in \mathbb{N}$. We call two t-tuples $(X_i)_{i=1}^t, (Y_i)_{i=1}^t$ compatible ($X \underset{0}{\sim} Y$) if their corresponding multisets (by forgetting the order of their elements), i.e. $X \underset{0}{\sim} Y$ for $X = L_t\left((X_i)_{i=1}^t\right), Y = L_t\left((Y_i)_{i=1}^t\right)$ defined by:

$$L_t : (2^E)^t \to (2^E \to \mathbb{N}_0), V \mapsto \sum_{i=1}^t \mathbb{1}_{\{V_i\}} \tag{2}$$

are compatible, and we have that $|X_i| = |Y_i|$ for each $i \in [t]$. (this automatically ensures that Equation 1 holds) ■

Lemma 4.1.2.2: Compatibility ($\underset{0}{\sim}$) is an equivalence relation between multisets and between tuples.

Proof: Compatiblity is an equivalence relation on the image of the multiset union.

Let E be a finite set. Let $X, Y, Z : 2^E \to \mathbb{N}_0$ be arbitrary multisets on 2^E.

Reflexivity: X is compatbile to itself ($X \underset{0}{\sim} X$) because the two equations have equal left- and right-hand sides if $X = Y$.

Symmetry: If $X \underset{0}{\sim} Y$, then $Y \underset{0}{\sim} X$ by swapping the left- and right-hand sides of the two equations.

Transitivity: If $X \underset{0}{\sim} Y$ and $Y \underset{0}{\sim} Z$, then it holds that $X \underset{0}{\sim} Z$ by combining all four input equations, yielding the expected result.

And for tuples, we first consider the map L_t (Equation 2) which maps tuples to multisets, for which reflexivity, symmetry and transitivity hold analogously. Then, for "$|X_i| = |Y_i|$ for each $i \in [t]$", we have $|X_i| = |X_i|$ (reflexivity), $|X_i| = |Y_i| = |X_i|$ (symmetry) and $|X_i| = |Y_i| \wedge |Y_i| = |Z_i| \Rightarrow |X_i| = |Z_i|$ (transitivity). □

Remark 4.1.2.3: Let E be a finite set (called the "ground set"). Let $t \in \mathbb{N}$. We can alternatively formulate compatibility of t-tuples as follows:

We call two t-tuples $(X_i)_{i=1}^t, (Y_i)_{i=1}^t$ compatible ($X \underset{0}{\sim} Y$) if all of the following hold:

$$\sum_{i=1}^t \mathbb{1}_{X_i}(e) = \sum_{i=1}^t \mathbb{1}_{Y_i}(e) \quad \forall e \in E$$
$$|X_i| = |Y_i| \quad \forall i \in [t]$$

■

The overall goal of all "Neil White conjectures" is to transform one t-tuple or multiset into another one, under the condition that they're compatible, and all their elements are bases of a given matroid, and transform them by means of symmetric (sometimes: multiple) matroid basis-exchanges.

Originally, in [7, p. 82], we also have the additional restriction that the exchange should be unique, in the sense that for a given choice of exchanging subset $A \subseteq X_m$ with $B \subseteq X_n$, there should only exist a single B such that the result $(X'_m \leftarrow (X_m \setminus A) \cup B, X'_n \leftarrow (X_n \setminus B) \cup A)$ is still a tuple of bases.

Definition 4.1.2.4: For the Neil White conjecture, there are various formulations that vary in how strict they are. We define the following equivalence relations:

Two compatible t-tuples are considered equivalent (under transitive closure of the following) if and only if they are the result of:
- $\underset{1}{\sim}$ a symmetric exchange between (Theorem 4.1.1.1) neighboring bases;
- $\underset{2}{\sim}$ a symmetric exchange between bases in the same tuple, but also allowing reordering;
- $\underset{3}{\sim}$ a multiple symmetric exchange (Lemma 4.1.1.2) between bases in the same tuple.

Denote by $\underset{1}{\sim}', \underset{2}{\sim}', \underset{3}{\sim}'$ the relations above before the taking of transitive closure. ■

4.2. The 2-tuples case with multiple symmetric exchanges

Let $M = (E, \mathcal{F})$ be a matroid.

Definition 4.2.1: Let $A, B \in \mathcal{F}$. The set of **multiple symmetric exchanges** between A and B is:

$$\{(A \triangle C, B \triangle C) \in \mathcal{F}^2 : (C \subseteq A \triangle B) \land (|A| = |A \triangle C|) \land (|B| = |B \triangle C|)\}$$

■

Note that (A, B) and (B, A) are contained in the set of all multiple symmetric exchanges between A and B.

Lemma 4.2.2: The set of all 2-tuples compatible to a given 2-tuple $(A, B) \in \mathcal{F}^2$ is equal to the set of *all multiple symmetric exchanges* between A and B.

Proof: Let $A, B, C, D \in \mathcal{F}$. Consider the tuples (A, B) and (C, D). We have that $|A| = |C|$ and $|B| = |D|$ if $(A, B) \underset{0}{\sim} (C, D)$ or (C, D) is in the set of all multiple symmetric exchanges between A and B.

Using Remark 4.1.2.3, we get that for $t = 2$ the set of equations for equal multiset union is then:

$$\sum_{e \in V \subseteq E} \mathbb{1}_{\{A\}}(V) + \mathbb{1}_{\{B\}}(V) = \sum_{e \in V \subseteq E} \mathbb{1}_{\{C\}}(V) + \mathbb{1}_{\{D\}}(V) \quad \forall e \in E$$

which can be further simplified into:

$$\mathbb{1}_A(e) + \mathbb{1}_B(e) = \mathbb{1}_C(e) + \mathbb{1}_D(e) \quad \forall e \in E$$

It follows that $A \cap B = C \cap D$ and $A \triangle B = C \triangle D$. If the multiset union is equal, then there exists an $F \subseteq A \triangle B$ such that $(C, D) = (A \triangle F, B \triangle F)$ by:

$$\begin{aligned} A \triangle B &= C \triangle D \\ A &= C \triangle D \triangle B \\ F = C \triangle A &= D \triangle B \end{aligned}$$

Let $F \subseteq A \triangle B$. We want to prove that the multiset union is then always equal. If $(C, D) = (A \triangle F, B \triangle F)$, then for each $e \in E$ it holds that:

$$
\begin{aligned}
& \mathbb{1}_F(e) = \mathbb{1}_F(e) \\
\Leftrightarrow \quad & \mathbb{1}_F(e) = \mathbb{1}_F(e)\mathbb{1}_{A \triangle B}(e) \\
\Leftrightarrow \quad & \mathbb{1}_F(e) = \mathbb{1}_F(e)(\mathbb{1}_A(e) + \mathbb{1}_B(e)) \\
\Leftrightarrow \quad & \mathbb{1}_F(e) = \mathbb{1}_{A \cap F}(e) + \mathbb{1}_{B \cap F}(e) \\
\Leftrightarrow \mathbb{1}_A(e) + \mathbb{1}_B(e) &= \mathbb{1}_A(e) + \mathbb{1}_F(e) - 2 \cdot \mathbb{1}_{A \cap F}(e) + \mathbb{1}_B(e) + \mathbb{1}_F(e) - 2 \cdot \mathbb{1}_{B \cap F}(e) \\
\Leftrightarrow \mathbb{1}_A(e) + \mathbb{1}_B(e) &= \mathbb{1}_{A \triangle F}(e) \qquad + \mathbb{1}_{B \triangle F}(e) \\
\Leftrightarrow \mathbb{1}_A(e) + \mathbb{1}_B(e) &= \mathbb{1}_C(e) \qquad + \mathbb{1}_D(e)
\end{aligned}
$$

□

4.3. Reduction of the t-tuples case to the 2-tuples case

We want to apply the result of Theorem 3.8 to the Neil White conjecture. So first, we need to pin down how to translate the important parts.

We want to note that the Neil White conjecture is equivalent to saying that $\underset{0}{\sim}$ and (depending on strength choice) some $\operatorname{mktup}_t \sim' \subseteq \operatorname{mktup}_t \underset{0}{\sim}$ have the same graph components (meaning that when representing each graph components as a set of nodes, those sets are equal; see also Remark 3.1). Also, mktup_t translates a graph where edges correspond to (multiple) symmetric exchanges for tuples of length 2 into the corresponding graph for tuples of length $t \geq 2$.

Let $M = (E, \mathcal{F})$ be a matroid. Let $\mathcal{B}$ be the basis system of M.

Theorem 4.3.1: Let $\sim' \subseteq \left(\underset{0}{\sim} \cap \binom{\mathcal{F}^2}{2}\right)$.

If the Neil White conjecture for multiple symmetric exchanges for tuples of length t holds[6], and we have that $\sim'$ has the same graph components as $\underset{0}{\sim}$ for $t = 2$,

then $\operatorname{mktup}_t \sim'$ has the same graph components as $\underset{0}{\sim}$.

Proof: Set (in Section 3) $\mathcal{F} \leftarrow \mathcal{B}, E \leftarrow \underset{0}{\sim}$. There are edges between all compatible tuples. Set $S \leftarrow \left(\underset{0}{\sim} \setminus \sim'\right)$.

Because $\sim'$ has the same graph components as $\underset{0}{\sim}$ for $t = 2$, this also holds for $\sim' = \underset{0}{\sim} \setminus \left(\underset{0}{\sim} \setminus \sim'\right) \subseteq \underset{0}{\sim}$.

By Theorem 3.8 it follows that for tuples of length $t \geq 2$, here $\operatorname{mktup}_t \underset{0}{\sim}$ also has the same graph components as $\operatorname{mktup}_t\left(\underset{0}{\sim} \setminus \left(\underset{0}{\sim} \setminus \sim'\right)\right) = \operatorname{mktup}_t \sim' \subseteq \operatorname{mktup}_t \underset{0}{\sim}$ because otherwise, it would have at least one additional graph component, meaning that two nodes which were connected before aren't anymore, but from that it follows that there must now also be two nodes in the graph of 2-tuples which aren't connected anymore, which is a contradiction to the assumption that the graph components for 2-tuples before and afterwards are equal.

And by the assumption that $\underset{0}{\sim}$ has the same graph components as $\operatorname{mktup}_t \underset{0}{\sim}$, we have the result. □

5. Notation

∧ logical and

[6] which is equivalent to saying that $\underset{0}{\sim}$ has the same graph components as $\operatorname{mktup}_t \underset{0}{\sim}$

$\vee$ logical or
$\cap$ set intersection
$\cup$ set union
$\uplus$ disjoint set union
$\triangle$ symmetric set difference $X \triangle Y := (X \setminus Y) \cup (Y \setminus X)$
$\subseteq$ subset (or equal) of another set, i.e. $X \subseteq Y \Leftrightarrow X \cap Y = X$
$[t]$ natural numbers up to $t : \mathbb{N}_0$, defined as $[t] := \{1, ..., t\}$
2^X for a given set X, 2^X denotes the powerset of X
$\binom{X}{k}$ for a given set X and $k \in \mathbb{N}_0$, $\binom{X}{k}$ denotes the set of all k-element subsets of X
$\binom{n}{k}$ for $n, k \in \mathbb{N}_0$, $\binom{n}{k} := \left|\binom{[n]}{k}\right| = \frac{n!}{k!(n-k)!}$
$\exists$ object exists (e.g. in a set)
$\forall$ for all objects (e.g. in a set)

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