

# Discussion about control of the heat equation

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## 1. Introduction

We have given the non-dimensionalized PDE for  $w \in H^2([0, 1] \times [0, \text{inf}))$

$$\begin{aligned}w_t &= w_{xx} \\w(0, t) &= w(1, t) := 0 \quad \forall t \geq 0 \\w(x, 0) &:= w_0(x)\end{aligned}$$

## 2. Lyapunov-Analysis

We want to inspect the following Lyapunov function:

$$\begin{aligned}V_1(t) &:= \frac{1}{2} \|w(\bullet, t)\|_{H^1}^2 \\V_1'(t) &= - \int_0^1 (w_x(x, t)^2 + w_{xx}(x, t)^2) dx = - \|w_x(\bullet, t)\|_{H^1}^2\end{aligned}$$

Note that we can omit the boundary terms during partial integration due to our boundary conditions.

We want to bound  $V_1'$  from above by  $V_1$  times some constant (and we want to get a somewhat optimal constant (i.e. large) because this affects our rate of convergence). So our goal is to find a relation between  $\|w(\bullet, t)\|_{H^1}$  and  $\|w_x(\bullet, t)\|_{H^1}$ .

More specifically, we want, for an  $g \in H_0^2([0, 1])$ , a relation between  $\|g\|_{H^1}$  and  $\|\dot{g}\|_{H^1}$ . For this, we use the Poincaré inequality, which applies not just to continuously differentiable functions, but to “weak functions” in  $L^p$  as well.

**Corollary 2.1:** Let  $\Omega$  be a bounded interval of  $\mathbb{R}$  with diameter  $|\Omega| = 1$ . Suppose that  $u : \Omega \rightarrow \mathbb{R}$  lies in  $H_0^1(\Omega)$  and the trace of  $u$  on the boundary  $\partial\Omega$  is zero. Then:

$$\|u\|_{L^2(\Omega)} \leq \pi^{-1} \|\dot{u}\|_{L^2(\Omega)}$$

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Proof for the Poincaré constant in the case of  $\Omega = [0, 1]$  can be found at <https://math.stackexchange.com/questions/360670/estimating-poincare-constant-for-unit-interval>. The constant can be found by solving a Sturm-Liouville problem.

Thus we get, by applying the corollary above twice:

$$\begin{aligned}V_1'(t) &= - \|w_x(\bullet, t)\|_{H^1}^2 \\&\leq -\pi \|w(\bullet, t)\|_{H^1}^2 = -2\pi V_1(t)\end{aligned}$$

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